A Genesis of Natural Numbers

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"*The origin is the goal*". Karl Kraus.

Abstract. We present here some ideas to change arithmetical teaching, from kinder garden up to tertiary math education. Beginning with syntax and semantics of numerical language, we introduce a new definition of natural numbers in order to construct equivalence classes of polynomials that change completely the concepts of factoring and primality of natural numbers. The most important thing is the introduction of a new procedure to check primality and to find prime factors for given, at least for now, small natural numbers.

Resumen. Presentamos en este trabajo, algunas ideas encaminadas a cambiar la enseñanza de la aritmética, desde el jardín infantil hasta la enseñanza universitaria. La inclusión de la sintaxis y la semántica del lenguaje numérico, nos permite lograr una nueva definición de número natural y construir clases de equivalencia de polinomios, que cambian radicalmente los conceptos de factorización y primalidad en los números naturales. Lo más importante hasta aquí, es, la introducción de un nuevo procedimiento para chequear primalidad, junto a un algoritmo para hallar los factores primos, al menos hasta ahora, de números naturales pequeños.

Introduction

Natural numbers are an important part of human culture. Number theory begins in Mesopotamia, following its path through Hellenistic times to merge in the Middle Ages with the current originated in Arab and Indian cultures. Leonardo Fibonacci left for us his mathematical mark in *Liber Abaci* (1202) mixing on it, hindu-arabic math with classic Greek number theory. From those times, number theory has been growing with the works of great mathematicians like Fermat, Euler, Gauss, Hilbert and many more.

Since antiquity natural numbers have been taken by granted, as Kronecker onetime said: "God made the integers, all the rest is the work of man". However, in the last part of nineteen century, Gottlob Frege begun a formalization of number theory introducing for first time a definition of number, based on set theory, a topic studied, among others, by Cantor.

Frege definition of number is rooted on the function set concept, namely, the association of a natural number to a set, in other words, the number of elements of that set. So a natural number, *n*, is the set of all sets where each set has *n* elements. From Frege's time, up to now, we have accepted this definition; nevertheless, much questioning about that did arise ever after. Frege's interest in defining number was to get a solid base in order to formalize number theory and to get proofs for properties, which were accepted before, as axioms relating number operations.

Here we start with some elementary notions: a vocabulary and a couple of operations defined with it: multiplication with the property of transmitting factoring and addition with the capacity of creating primality. The reiterative use of these operations, together with some simple syntax rules, give rise to some entities called here polynomials.

The Origin

We don't know exactly in what period of human history, numbers, as we understand today, appear. However, we think, that some operations or routines preceded numbers. A routine operation as adding, doesn't need to know numbers, it is just putting together objects to perform a new set; also, multiplication could be seen, as a repetition of previous addition processes. The first approximation to the abstract concept of number could be through the natural process of counting.

Counting is present in most primitive cultures from the past, up till now. Language made counting something important because of the possibility to transmit information and to have at hand methods to make the most primitive calculations. In primitive times, we can imagine, human beings doing additions through handling small objects and multiplications through repetitions.

So, let's try to replay this ancient processes starting with addition and multiplication in their most primitive meaning, adding as putting together and multiplying as repeating. Language contributes with symbol words to ease, all of these elementary procedures. Imagine we have previously established a grouping pattern, say ten; and we have number words for 0, 1, … , 9, then we can go further with ten, with the meaning of one package of ten units. Here appears the great jump from addition to multiplication. A ten package really means, one, counted ten times. From here up begins the combination of grouping and repeating.

Suppose again, we just have now numbers words and not yet number symbols. The number word *eleven*, would have the meaning one ten plus one; in the same vein, twelve would be, one ten plus two, fifty eight, would have a meaning of five (times) ten plus (and) eight, and so up to ninety nine as nine (times) ten plus (and) nine. In one hundred there is another jump because we put together ten packages of ten to perform a major unity, namely, one hundred to mean ten times ten. This primitive process, human beings, come repeating over and over again throughout millennia, changing of course the pattern of counting, in some cases ten, twelve, twenty, even, sixty.

So, in the beginning was language. Number symbols are very recent invention of the humans. There are no records of number symbols, say, before fifty thousand years ago and we can suppose human race is living in the world for around two hundred thousand years. Inside language there is a lot of information that, we, as teachers, may use as tools for improving basic mathematics teaching. The early use of numerical language in elementary school, is one the aims of our math education project.

Emphasizing on numerical representation through language is the most important thing in teaching elementary math for kids. Before using number symbols, children have to be able to understand number words, their syntax and semantics. When they hear the words "*forty three*", they understand "forty and three, or, forty plus three" and can translate to a real situation of "four packs of ten and three more units", or, semantically: "*four times ten plus three*". This last sentence, carry on, in its meaning, a profound information message: not only a numerical information, also there is a syntax on it, together with a meaning (semantics). The arrangement of words have an order, and the terms, "times" and "plus", are associated to the most elementary concepts of addition and multiplication.

According to our experience kids assimilate early and easily numbers words together with its syntax and semantics. At kinder garden they get in contact with number symbols and are approaching to adding and multiplication processes. At this stage of formation, the most important aspect to take in account is the use of their hands and, contrary to the customs; they may touch anything they can, of course, except those things that might have any risk for their security.

We propose to reach natural numbers by following the path:

1) Creating motivation toward the ability to handle small objects to make piles, groups, lots, assortments, and any kind of collections with special characteristics. The goal here is getting the natural mechanism of adding.

2) To make some selection of a sort of pattern, for instance: pairs, triplets, quadruples, quintuples, etc. Repeating experiences of this type, we are motivating the child toward the comprehension of multiplication.

The idea behind these first experiences, is to pave the road to introduce natural numbers in a naïve way using addition and multiplication, as primitive elements. Since adding and multiplying is so basic to the apprehension of the number concept, these two operations has to be a most, at kinder garden preparation to define and understand the number concept. These two operations are the deep rock where polynomials will stand. From some special type of polynomials, we are going to construct, all arithmetical theory to be taught at elementary school.

<u>The Number Language</u>.

Along the way we are introducing number words, we are slowly replacing numbers words by number symbols. The first step, of course, would be, showing the most simple symbols to write the smallest numbers, namely, the *digit* set, $S = \{0,1,2,3,4,5,6,7,8,9\}$, equivalent to an alphabet for writing all natural numbers, whenever we need a decimal representation. We also take some primitive parameters like "+" and " · "[1]. When we begin using mathematical symbols (*numerals*), in our language, we can say that we are changing from colloquial language to numerical language.

The words in numerical language are strings of symbols, where the alphabet digits are linked by the primitive parameters, "+", and, " · ", with the syntactic rule,

$$x^n = 10x^{n-1} \qquad (*)$$

The meaning of (*) in our context, is: the value of n-th position inside the numeral, counted from 0 to n and from right to left, is ten times the value of (n-1)-th position in the numeral. That is the reason why our number system is a decimal positional system. If we would use the syntactic rule, $x^n = 2x^{n-1}$, our numerical system would be a binary one, with the alphabet {0, 1}. Let's see a couple of examples.

Example 1

The number, *four thousand three hundred twenty one*, given in colloquial language means in numerical language "four times thousand, plus, three times hundred, plus, two ten, plus, one" and its symbolic representation is;

$$4321 = 4 \cdot 1000 + 3 \cdot 100 + 2 \cdot 10 + 1$$

We see clearly, how are arranged the ten powers, in descending order from 10^3 down to 10^0 = 1. In this number representation, the syntactic rule (*) is explicitly shown, meaning that left digit corresponds to thousands, the following to hundreds, at right tens and finally the units. Using powers of ten and changing 10 by x, we get

$$4321 = 4 \cdot 1000 + 3 \cdot 100 + 2 \cdot 10 + 1 = 4 \cdot 10^3 + 3 \cdot 10^2 + 2 \cdot 10 + 1 \cdot 10^0 = 4 \cdot x^3 + 3 \cdot x^2 + 2 \cdot x + 1$$

The last expression, remind us the one variable polynomial, $P(x) = 4x^3 + 3x^2 + 2x + 1$, with x arranged in descending order. Polynomials like these, we call *standard decimal polynomials*. This kind of polynomials is the base for our main definition below.

Example 2

A number like 987, as any other, can be written with the use of (*), in several ways, besides the standard form described above. Let's show some of them:

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[1] All machinery inherent to formal languages, is suggested at: http://www.matematicasyfilosofiaenelaula.info/articulos/Sintax_and_Semantics_of_Numerical_Lenguage_at_Elementary_School.pdf

$987 =$ (*standard form*) $9x^2 + 8x + 7 = 8x^2 + x^2 + 8x + 7 = 8x^2 + 10x + 8x + 7 = 8x^2 + 18x + 7 = 8x^2 + 17x + x + 7 = 8x^2 + 17x + 17 = 7x^2 + 27x + 17 = 6x^2 + 38x + 7 = \ldots$

If we replace x by 10 we recapture the initial number 987.

Factoring and Primality

Factoring is the property of a number of being expressed as a product of two or more factors. As main definition below accepts, any number is either prime or composite. However a prime number p, has also the property $p\cdot 1 = 1\cdot p = p$. So multiplication preserves factors anyhow the number be either prime or composite. For the case of composite numbers, this property is more visible. For instance, the prime factors of 6 are 2 and 3 and the prime factors of 35 are 7 and 5, we can easily check that the prime factors of $6\cdot 35 = 210$ are: 2, 3, 5 and 7, the same factors of 6 and 35. In general, if we multiply any number $n \neq 1$, by $m \neq 1$, the product $n\cdot m$ is a composite number.

Addition, however, has the property of transforming primes in composite or vice versa, composite numbers in primes. Not always, of course, but we can. Take for example any prime $p \neq 2$, then $p + 1$ is not prime because, all primes, except 2, are odd and so $p + 1$ is even and so, it is a multiple of 2. Also, take a composite as 12 and add to it 7, to get the prime 19. When we add two prime numbers we are not sure, either we are going to get a prime or a composite number, as in the following cases: $2 + 3 = 5$ (*prime plus prime gives prime*); $3 + 7 = 10$ (*prime plus prime gives composite*). In general, (naively) every even integer greater than two, can be expressed as a sum of two primes (Goldbach's conjecture).

A natural number a, written with its own digits $a_n, a_{n-1}, \ldots, a_1, a_0$, as $(a_n a_{n-1} \ldots a_1 a_0)$, looks like:

$$a = (a_n a_{n-1} \ldots a_1 a_0) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 x^0 = \sum_{j=0}^{j=n} a_{n-j} x^{n-j}$$

When $x = 10$ we recover the conventional form of representing $a = (a_n a_{n-1} \ldots a_1 a_0)$. We call the above polynomial, **the standard polynomial representation** of a. Note that the coefficients in this polynomial, are the same digits as in the decimal representation of a.

Definition 1.We say that two polynomials $P(x)$ and $Q(x)$, are equivalent module a, if $P(10) = Q(10)$. If that is the case, we use the notation, $P(x) \equiv Q(x)$ (module a).

For instance, $a = 236$ can be associated to the standard polynomial $p(x) = 2x^2 + 3x + 6$, where the coefficients of p are the digits 2, 3, 6. Using syntactic rule (*) on p, we get an equivalent polynomial such as, $q(x) = x^2 + 13x + 6$, so, $2x^2 + 3x + 6 \equiv x^2 + 13x + 6$ (module a), since, p

(10) = q (10). Easily we can check that: $10^2 + 13(10) + 6$, and, $2(10)^2 + 3(10) + 6$, give the same result: 236.

The main definition below, gives us a different way to see natural numbers as equivalent classes of one variable polynomials. For each number a, we can construct a family of equivalent polynomials, via the syntactic rule (*).

The Main Definition

We define number $a = (a_n a_{n\text{-}1} \ldots a_1 a_0)$, as the family of all equivalent polynomials module a as:

$\Omega_a = \{ \sum_{j=0}^{j=n} b_{n-j} x^{n-j}$, such that, any polynomials p, q, in the family satisfies $p(10) = q(10)\}$.

The classical definition of prime number is linked to the concept of division. Here we take another road: we use factoring.

Definition 2

A number a, defined using Ω_a, is said to be *composite* if there exists a polynomial p in Ω_a, which can be factored. Otherwise a will be called *prime*.

Example 3. Find the prime factors of $a = 5893$.

The equivalence class for this number is:

$\Omega_a = \{5x^3 + 8x^2 + 9x + 3,\ 50x^2 + 8x^2 + 9x + 3,\ 58x^2 + 9x + 3,\ 57x^2 + x^2 + 9x + 3,\ 57x^2 + 10x + 9x + 3,\ 57x^2 + 19x + 3,\ 56x^2 + 29x + 3,\ \ldots,\ 5893\}$

The polynomial, $56x^2 + 29x + 3$ in Ω_a, can be factored in the following way:

$56x^2 + 29x + 3 = (7{\cdot}8)x^2 + (7{\cdot}3)x + (8)x + 3 = 7x(8x + 3) + (8x + 3) = (7x + 1)(8x + 3)$.

When we replace $x = 10$ the first polynomial gives 5893 and the last one gives $71{\cdot}83$ and so, $5893 = 71{\cdot}83$.

This example show us, how in some cases, it is easy to find the prime factors of a composite number without using division at all. Inside the class Ω_a we may see all its elements as equals and operate them as polynomials, having all their properties, among others, and the most important one: the *unique factorization property*.

Example 4. In classical number theory, the way to decide, if either a number a is prime or composite, is through the repeated division of a by the primes p, $2 \leq p \leq \sqrt{a}$. Here, we make use of main definition for a, instead. We only check Ω_a, for a factorable polynomial. Let's check the number $a = 127$ for primality.

$\Omega_a = \{x^2 + 2x + 7,\ 10x + 2x + 7,\ 12x + 7,\ 127\}$

All polynomials in Ω_a are equivalent module 127, i.e., when we change x by 10 we get 127. If one of them factors, the number 127, also factors, otherwise, 127 is prime. No polynomial in Ω_a factors, then 127 is a prime number.

Until here, the problem to decide if a is or not prime have been reduced to factoring quadratic polynomials. To practice the techniques of factoring polynomials I suggest some of my past works about this topic[2].

Example 5. Finding factors of number a = 15347 takes a little more time. With some shortcuts we get:

$\Omega_a = \{x^4 + 5x^3 + 3x^2 + 4x + 7,\ 153x^2 + 4x + 7,\ 152x^2 + 14x + 7,\ 151x^2 + 24x + 7,\ 150x^2 + 34x + 7,\ 149x^2 + 44x + 7, \ldots, 15347\}$

We can factor:

$149x^2 + 44x + 7 = 149x^2 + 440 + 7 = 149x^2 + 447 = 149x^2 + 3\cdot 149 = 149(x^2 + 3) = 103\cdot 149.$

We replaced above, $447 = 44x + 7 = 4x^2 + 4x + 7 = 3(x^2 + 4x + 9) = 3\cdot 149$

That shows that $15347 = 103\cdot 149$.

In this paper I have tried to show another way to see natural numbers, and also, how we can change traditional routines for rational algorithms, easily comprehensible to kids, I hope.

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[2]See: http://www.matematicasyfilosofiaenelaula.info/articulos/Numbers_as_%20a_product_of_Primes.pdf
And:
http://www.matematicasyfilosofiaenelaula.info/conferencias/The_Number_Language_eimat2015_Conferencia.pdf